Local, Smooth, and Consistent Jacobi Set Simplification

Harsh Bhatia
hbhatia@sci.utah.edu
SCI Institute, University of Utah

Bei Wang
beiwang@sci.utah.edu
SCI Institute, University of Utah

Gregory Norgard
gregnorgard@gmail.com
Numerica Corporation

Valerio Pascucci
pascucci@sci.utah.edu
SCI Institute, University of Utah

Peer-Timo Bremer
bremer5@llnl.gov
Lawrence Livermore National Laboratory

Abstract

The relation between two Morse functions defined on a common domain can be studied in terms of their Jacobi set. The Jacobi set contains points in the domain where the gradients of the functions are aligned. Both the Jacobi set itself as well as the segmentation of the domain it induces have shown to be useful in various applications. Unfortunately, in practice functions often contain noise and discretization artifacts causing their Jacobi set to become unmanageably large and complex. While there exist techniques to simplify Jacobi sets, these are unsuitable for most applications as they lack fine-grained control over the process and heavily restrict the type of simplifications possible.

In this paper, we introduce a new framework that generalizes critical point cancellations in scalar functions to Jacobi sets in two dimensions. We focus on simplifications that can be realized by smooth approximations of the corresponding functions and show how this implies simultaneously simplifying contiguous subsets of the Jacobi set. These extended cancellations form the atomic operations in our framework, and we introduce an algorithm to successively cancel subsets of the Jacobi set with minimal modifications according to some user-defined metric. We prove that the algorithm is correct and terminates only once no more local, smooth and consistent simplifications are possible. We disprove a previous claim on the minimal Jacobi set for manifolds with arbitrary genus and show that for simply connected domains, our algorithm reduces a given Jacobi set to its simplest configuration.

1 Introduction

In scientific modeling and simulation, one often defines multiple functions, e.g. temperature, pressure, species distributions etc. on a common domain. Understanding the relation between such functions is crucial in data exploration and analysis. The *Jacobi set* [4] of two scalar functions provides an important tool for such analysis by describing points in the domain where the two gradients are aligned, and thus partitioning the domain into regions based on relative gradient orientation. A variety of interesting physical phenomena such as the interplay between salinity and temperature of water in oceanography [1] and the critical paths of gravitational potentials of celestial bodies [18] (similar to the Lagrange points in astrophysics) can be modeled using Jacobi sets. In data analysis and image processing, Jacobi sets have been used to compare multiple scalar functions [6], as well as to express the paths of critical points overtime [3, 4], silhouettes of objects [8], and ridges in image data [16].

However, the Jacobi sets can be extremely detailed to the point at which their complexity impedes or even prevents a meaningful analysis. Often, one is not interested in the fine-scale details, e.g. minor silhouette components due to surface roughness, but rather in more prevalent features such as significant protrusions. The Jacobi sets are also highly sensitive to noise which further leads to undesired artifacts. Finally, the most common algorithm to compute Jacobi sets [4, 16] is designed for piecewise linear functions defined on triangulations, and is well known to introduce a large number of discretization artifacts which could skew the analysis. The natural answer to these problems is the controlled simplification of a Jacobi set by ranking and ultimately removing portions of it in order of importance.

Some previous techniques exist that can be broadly classified into direct and indirect Jacobi set simplification. *Indirect* simplification [3, 9] simplifies the underlying functions in a hope to obtain a structurally and geometrically simpler Jacobi set. However this poses several problems. First, especially in the case of two non-trivial functions, changing either of them can introduce a large number of complex changes in the Jacobi set. These changes are difficult to predict and track, and instead the Jacobi set is typically re-computed at each step, which quickly becomes costly. Second, the Jacobi set encodes the relation between two functions and therefore simplifying one function may not actually simplify the Jacobi set. For example, two functions with complex gradient flows, which are similar in terms of relative orientation, define a small and simple Jacobi set. In this case, smoothing the gradient flow of either of the functions can introduce significant additional complexity into the Jacobi set. Finally, creating an appropriate metric to rank potential simplification steps can be challenging as small changes relative to traditional function norms e.g. L_2 or L_∞ may induce large changes in the Jacobi set and vice versa.

Alternatively, *direct* simplification aims to identify and remove "unimportant" portions of the Jacobi set and subsequently to determine the necessary changes in the corresponding functions. Such techniques are designed to reduce the complexity of a Jacobi set measured by a user-defined metric. The first step [14] proposed in this direction views the Jacobi set as the zero level set of a complexity measure [6] and removes components (i.e. loops) of the level set in order of their hyper-volume. However, this strategy is limited to removing entire loops of the Jacobi set. In practice, much of the complexity of the Jacobi set is due to small undulations in the level sets of the functions causing zig-zag patterns. Such features are not addressed directly by a loop removal, which severely limits the usability of this approach. Furthermore, as discussed in Section 3, one can easily construct cases where loops should be combined rather than removed. In contrast, our goal is to obtain a Jacobi set with fewer *birth-death* (BD) points (where the level sets of the two functions and the Jacobi curve have a common normal direction) and fewer loops.

Contributions To overcome the current limitations in Jacobi set simplification, we introduce a new direct simplification framework for Jacobi sets of two Morse functions defined on a common smooth, compact, and orientable 2-manifold without boundary. By extending the notion of critical point cancellations in scalar fields to Jacobi sets, we identify all possible simplifications that are realizable by smooth approximations

of the corresponding functions. Based on a user-defined metric, we then rank these operations and progressively simplify the Jacobi set until no further reduction is possible. Our framework provides a fine-grained control over a very general set of possible simplifications and allows, e.g. the combination of loops and the removal of zig-zag patterns along side the traditional loop removal. In particular:

- We introduce the notion of *local* pairings of points in the Jacobi set that can be cancelled. These point-wise cancellations are then extended to contiguous sub-domain bounded by segments of the Jacobi set, referred to as *Jacobi regions*, which are simplified simultaneously in a *consistent* manner. To obtain *smooth* realization of the simplification, the modification of Jacobi regions are extended to collection of adjacent regions, referred to as *Jacobi sequences*. Each such sequence is a contiguous sub-domain ranked by a user-defined metric and is simplified as one atomic operation;

- We propose a simplification algorithm that constructs and successively cancels Jacobi sequences. Our approach naturally cancels critical points of both functions, removes and combines loops, straightens the Jacobi set by removing zig-zag patterns, and always reduces the number of BD points;

- We show that the algorithm is correct, which means that the simplified Jacobi set is valid, and that it terminates only when no more local, smooth, and consistent simplifications are possible;

- We disprove a previous claim on the minimal Jacobi set for manifolds with arbitrary genus and show that for domains with even genus there always exist function pairs that create a single loop in the Jacobi set; and

- We show that for simply-connected domains, our algorithm reduces a given Jacobi set to its minimal configuration; while for non-simply-connected domains, we discuss some fundamental challenges in Jacobi set simplification.

2   Background and Related Work

This section presents the relevant background on Morse theory [10, 11] and Jacobi sets [4], and discusses the existing Jacobi set simplification schemes. In the following, let $\mathbb{M}$ be a smooth, compact, and orientable 2-manifold without boundary.

Morse functions   Given a smooth function $f : \mathbb{M} \to \mathbb{R}$, a point $x \in \mathbb{M}$ is called a *critical point* if the gradient ∇f of f at x equals zero, and the value of f at x is called a *critical value*. All other points are *regular points* with their function values being *regular values*. A critical point x is *non-degenerate* if the Hessian, i.e. the matrix of second partial derivatives at the point, is invertible. f is a *Morse function* if (a) all its critical points are non-degenerate and (b) all its critical values are distinct.

Jacobi set   Given two generic Morse functions $f, g : \mathbb{M} \to \mathbb{R}$ such that the intersection of the sets of their critical points is a null set, their Jacobi set $\mathbb{J} = \mathbb{J}(f, g) = \mathbb{J}(g, f)$ is the closure of the set of points where their gradients are linearly dependent [4],

$$\mathbb{J} = \mathrm{cl}\,\{x \in \mathbb{M} \mid \nabla f(x) + \lambda \nabla g(x) = 0 \text{ or } \nabla g(x) + \lambda \nabla f(x) = 0\}. \tag{1}$$

The sign of λ for each x is also called as its *alignment*, as it defines whether the two gradients are aligned or anti-aligned. By definition, the Jacobi set contains the critical points of both f and g. Let $g^{-1}(t)$ represent
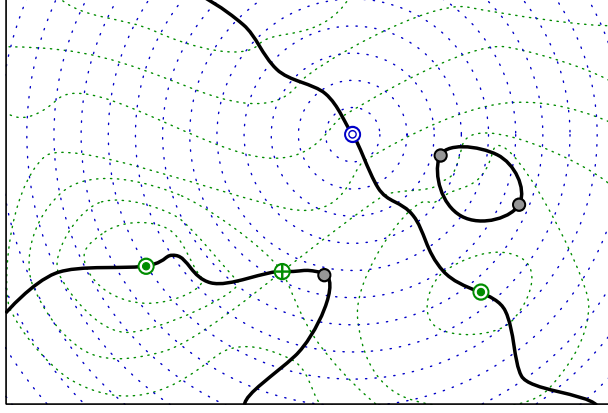
Figure 1: Jacobi set (solid black) of two functions with BD points shown in grey and critical points of the function in blue and green respectively.

the *level sets* of g for $t \in \mathbb{R}$, and $f_t : g^{-1}(t) \to \mathbb{R}$ the restriction of f on the level sets of g. Equivalently, Jacobi set can be defined as the closure of the set of critical points of f_t for all regular values t of g [4].

$$\mathbb{J} = \mathrm{cl}\left\{x \in \mathbb{M} \mid x \text{ is critical point of } f_t\right\}. \tag{2}$$

The critical points of f_t are also referred to as the *restricted critical points* of f (with respect to g). The restricted function f_t is a Morse function almost everywhere[1]. Three types of degeneracies exist where f_t is not Morse for some $t \in \mathbb{R}$: (a) t is a critical value of g, then the level set $g^{-1}(t)$ contains a singularity and thus is not a 1-manifold; (b) Two or more critical points in f_t share the same function value; (c) f_t contains an inflection point (a degenerate critical point). These degeneracies play an important role in our discussion on Jacobi set simplification. For example, each restricted critical point along $\mathbb{J}$ is an extrema of f_t for some $t \in \mathbb{R}$. As t varies, maxima and minima of f_t can approach each other and ultimately merge at an inflection point called a *birth-death (BD) point*. Alternatively, traveling along $\mathbb{J}$, restricted critical points of f_t switch their criticality (from maximum to minimum or vice versa) at BD points. Furthermore, the restricted functions f_t switch criticality at critical points of g (but not at critical points of f). Similarly, the alignment of restricted critical points switches at critical points of both f and g. Figure 1 illustrates these concepts.

Comparison measure   Several other descriptions of Jacobi sets exist [4, 5, 6, 14]. One such description is in terms of a gradient-based metric to compare two functions, called the *comparison measure* κ [6]. It plays a significant role in assigning an importance to subsets of a Jacobi set in terms of the underlying functions f and g by measuring the relative orientation of their gradients. For a domain Ω,

$$\kappa = \kappa(\Omega) = \frac{1}{\mathrm{Area}\,(\Omega)} \int_{x \in \Omega} \kappa_x \, \mathrm{d}x = \frac{1}{\mathrm{Area}\,(\Omega)} \int_{x \in \Omega} ||\nabla f(x) \times \nabla g(x)|| \, \mathrm{d}x,$$

where $\mathrm{d}x$ is the area element at x, and $\mathrm{Area}\,(\Omega) = \int_{x \in \Omega} \mathrm{d}x$. Here $\kappa_x = ||\nabla f(x) \times \nabla g(x)||$ represents the limit of $\kappa = ||\nabla f \times \nabla g||$ to a single point, and the Jacobi set is its 0-level set [6, 9, 14].

Level set neighbors   For a point $v \in \mathbb{J}$, we can define its *level set neighbors* (with respect to g), $n_g(v)$, as neighbors of $v \in \mathbb{J}$ along $g^{-1}(g(v))$. Two points $u, v \in \mathbb{J}$ are level set neighbors if $u \in n_g(v)$ which implies

---

[1]The set of points where the function f_t is not Morse is a finite set of measure zero.

$v \in n_g(u)$. Generically, $|n_g(v)| \leq 2$, however, for an extremum of g, $|n_g(v)| = 0$, and for a saddle of g, $|n_g(v)| \leq 4$. The level set neighbors can symmetrically be defined with respect to f. Such a definition can be extended to smooth curves in $\mathbb{J}$. Two smooth parametrized curves $\alpha, \beta : (a, b) \to \mathbb{M}$ in $\mathbb{J}$ are level set neighbors if $\alpha(t)$ and $\beta(t)$ are level set neighbors in $g^{-1}(t)$ for all $t \in (a, b)$. For simplicity in notations, for such level set neighbors, we choose a and b to be function values of g, i.e. $g(\alpha(a)) = a$ and $g(\alpha(b)) = b$. We further define their *bounded region*, denoted by $R_{(a,b)}(\alpha, \beta)$, as the open subset of $\mathbb{M}$ bounded by curves α, β, and level sets of g that pass through their end points, i.e. $g^{-1}(a)$ and $g^{-1}(b)$.[2]

Related Work

As discussed above, the Jacobi set may contain a number of components that represent noise, degeneracies, or insignificant features in the data. As a result, Jacobi set simplification is both necessary and desirable. Bremer et al. [3] use the Jacobi set to track the critical points of a time-varying function $f : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$, where time is represented as $g : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$ and $g(x, t) = t$. The Jacobi set $\mathbb{J} = \mathbb{J}(f, g)$ is therefore the trajectory of the critical points of f_t as time varies. To simplify the Jacobi set, they use the Morse-Smale complex of f_t at discrete time-steps to pair critical points, cancel pairs below a persistence threshold, and remove small components of the Jacobi set that lie entirely within successive time-steps. This method, however, is difficult to extend to a general setting: First, only one function, f, is simplified and the other is assumed to be trivial; Second, only a small, discrete number of f_t are simplified and all intermediate changes are ignored.

Luo et al. [9] propose an algorithm to compute the Jacobi set of a point cloud. The Jacobi set is considered as the 0-level set of κ_x, which is computed by approximating the gradients ∇f and ∇g. Reducing the number of eigenvectors used in the gradient approximation, therefore, corresponds to a simpler Jacobi set after re-computation. This is the foremost example of an indirect simplification in which f and g are smoothed which leads to some (unpredictable) changes in $\mathbb{J}$. Instead, as discussed below, this paper aims at identifying and removing an unimportant portion of $\mathbb{J}$ by determining how f and g can be modified accordingly.

N and Natarajan [14] consider the simplification of the Jacobi set as the reduction in the number of components in $\mathbb{J}$ with minimal change to the relationship between the two functions, quantified by κ_x. The authors construct the Reeb graph [17] of κ_x, and associate a percentage of κ as offset cost with each critical point and 0-level set point in the Reeb graph. A greedy strategy is then applied to modify a component in the Jacobi set with the least offset cost until a threshold is reached. However, this technique is restricted to removing entire loops of $\mathbb{J}$, which significantly restricts its flexibility. For example, one can easily construct examples where $\mathbb{J}$ is highly complex yet contains only a single loop.

3 Jacobi Set Simplification – An Overview

As discussed in Section 1, this paper introduces a direct simplification of the Jacobi set, i.e. it removes a given set of points from $\mathbb{J}$, by understanding the required changes in f and/or g. The goal is to obtain a Jacobi set with fewer BD points and/or fewer loops, by making the gradients of the underlying functions more similar. In the following, we describe simplification of $\mathbb{J}$ which modifies f with respect to the level sets of g, but all concepts apply symmetrically to modifications of g with respect to f. In practice, we consider simplifications that modify either f or g, and typically interleave operations acting on one or the other.

Since the Jacobi set $\mathbb{J}(f, g)$ is defined as the closure of the (restricted) critical points of f_t for regular values $t \in \mathbb{R}$ where f_t is a 1D function, it is natural to simplify $\mathbb{J}$ by canceling restricted critical points in f_t. In the topological simplification of a scalar function, typically the features of interest are critical points.

---

[2]In the case where $\alpha, \beta : (a, b) \to \mathbb{J}$ are subsets of some larger parametrized curves $\alpha', \beta' : (a', b') \to \mathbb{J}$, that is, α, β are the restriction of α', β' to $(a, b) \subseteq (a', b')$, i.e. $\alpha = \alpha'|_{(a,b)}$ and $\beta = \beta'|_{(a,b)}$, we denote their bounded region as $R_{(a,b)}(\alpha', \beta')$.

Based on the Morse Cancellation Theorem (Theorem 5.4 in [12] as the First Cancellation Theorem, or [13]), critical points must be removed in pairs through atomic cancellation operations. Therefore, we remove pairs of restricted critical points to construct continuous simplified function $\overline{f_t}$, such that no other critical points of f_t are affected. To obtain a smooth approximation f_t^* of the simplified function $\overline{f_t}$, the modification can be extended to allow an ϵ-slope for the modified function. The region of influence of this cancellation is the region where $f_t \neq f_t^*$, and is highlighted in Figure 2(a). In order to perform these cancellations, we must first define a scheme for pairing restricted critical points. Section 4.1 discusses in detail the choice of our pairing scheme, and the procedure of carrying out such cancellations.

Although cancellation of restricted critical points produces smooth simplified restricted functions f_t^* as shown in Figure 2(a), performing a single such cancellation creates a discontinuity across the level set $g^{-1}(t)$. In order to obtain smoothness across level sets, we must extend these cancellations by canceling more than one contiguous pairs of restricted critical points, called the *Jacobi regions*, at the same time. For example, consider two Jacobi regions V_1 and V_2 existing between the level sets $g^{-1}(a)$, $g^{-1}(c)$, and $g^{-1}(b)$ as shown in Figure 2(b) (left). These Jacobi regions represent contiguous pairs of restricted critical points shown as red and blue lines respectively. A smooth simplification f^* that cancels all critical points in these regions can be obtained by modifying f in the corresponding shaded region. The construction, properties, and cancellation of the Jacobi regions are discussed in detail in Section 4.2.

The cancellation of Jacobi regions, however, creates smooth functions only at the interior of the regions, and the discontinuities are pushed to their boundaries. In order to create globally smooth simplified functions f^*, we must further cancel a sequence of adjacent regions at the same time, e.g. canceling V_1 and V_2 at the same time as shown in Figure 2(b) (right). We show that any discontinuities can be avoided by local modifications if these *Jacobi sequences* start and end with BD points, and discuss their construction and cancellation in Section 4.3.

This way, the entire Section 4 focuses on a simplification scheme that extends the concept of critical point cancellation in scalar functions to Jacobi sets. The defining characteristic of a *valid simplification* is the removal of pairs of restricted critical points in $\mathbb{J}$ in a *local*, *smooth*, and *consistent* manner.

Definition 3.1 (Valid Simplification) Let V be a set of level set neighbors in $f_{[a,b]} = \{f_t \mid t \in [a,b]\}$ for some interval $[a,b] \subseteq \mathbb{R}$. Removing V from $\mathbb{J}$ is considered a *valid simplification* if it is

1. *local*: There exists a continuous $\overline{f}_{[a,b]} = \{\overline{f}_t \mid t \in [a,b]\}$ containing all critical points of $f_{[a,b]}$ except for V;
2. *smooth*: There exists a smooth $f^* : \mathbb{M} \to \mathbb{R}$ such that $||f_{[a,b]}^* - \overline{f}_{[a,b]}||_\infty < \varepsilon$ for any $\varepsilon > 0$; and
3. *consistent*: $\mathbb{J}(f^*, g) = \mathbb{J}(f, g)$ for all x with $g(x) \in (-\infty, a) \cup (b, \infty)$, and $f^*(x) = f(x)$ for all x with $g(x) \in (-\infty, a - \epsilon] \cup [b + \epsilon, \infty)$ for any $\epsilon > 0$.
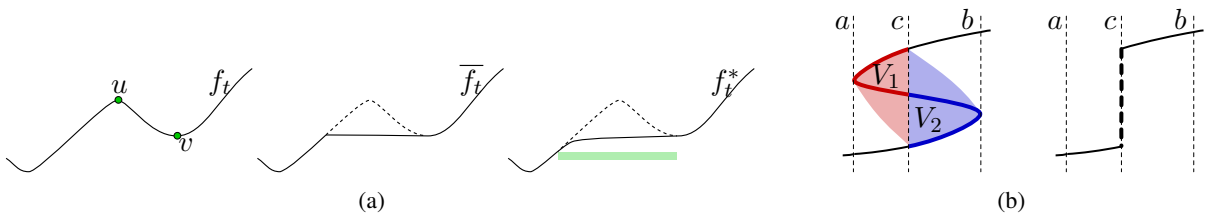

Figure 2: (a) Cancellation of a pair of critical points in the 1-dimensional f_t (left) gives a continuous simplified simplified $\overline{f_t}$ (middle). A smooth simplification can be obtained as f_t^*. The cancellation modifies f_t in the shaded region. (b) (Left) Valid simplification of V_1 and V_2 must construct smooth f^* by modifications in the corresponding shaded regions only. a, b, and c represent level sets of g. (Right) The simplified Jacobi set.

We point out that the simplified function $f^*(x)$ is defined with respect to a given ϵ. Referring to Figure 2(a) it is important to note that the locality conditions implies that the modification in f_t must not impact any restricted critical points other than u and v. In Figure 2(b), this means that for any level set of g (vertical line), the red and blue shaded regions must not touch any portion of Jacobi set other than the ones shown in red and blue respectively. Notice that while locality is associated with continuous function $\overline{f_t}$, the second condition requires f^* to be smooth along and across level sets. In order to create such a smooth f^*, the locality condition must be relaxed within a small neighborhood around the cancelled Jacobi region. Furthermore, the consistency condition requires that no portions of Jacobi set outside $[a, b]$ are modified. In a way, the consistency condition implies locality across level sets. While the locality condition is obtained by defining a special pairing function, a smooth and consistent simplification can be performed when the Jacobi sequence begins and ends with BD points (as in Figure 2(b)). Also, notice that $\mathbb{J}(f^*, g) \not\subset \mathbb{J}(f, g)$, since new points (dashed line) may be added to the Jacobi set to connect the existing curves.

Unfortunately, as detailed in Section 4.3, the saddles of g present unresolvable discontinuities in the pairings, and therefore may obstruct the construction of Jacobi sequences. Consequently, the simplification scheme discussed above may not be able to progress. In order to handle such cases, we use a conventional critical point cancellation technique in 2D to cancel a saddle of g with its maxima/minima. As shown in Section 5, our approach does not change the Jacobi set structurally, but only simplifies (reduces the number of) its alignment switches.

Using the simplification techniques discussed in Sections 4 and 5, Section 6 presents a combined procedure for simplifying Jacobi sets which can be guided by an arbitrary metric. We provide correctness proofs for the procedure, and show that for simply-connected domains, this procedure obtains the simplest possible configuration of Jacobi sets. On the other hand, for non-simply-connected domains, we discuss current challenges and list them as future work.

4 Cancellation of restricted critical points in f

This section details the procedure of canceling restricted critical points in $\mathbb{J}$ to obtain simplified functions. Starting with the simplification of 1D restricted functions, we discuss the cancellation of entire segments of $\mathbb{J}$ by canceling Jacobi sequences.

4.1 Pairing and cancellation of restricted critical points

Since restricted critical points of f_t must be cancelled in pairs, we need a mechanism to define such pairings. The *topological persistence pairing* [7, 19] seems to be an obvious choice, where critical points are paired and removed in order of *persistence*. However since persistence pairing is assigned globally, restricted critical points, which are not level set neighbors may be paired. These pairs cannot be cancelled without violating the locality condition, which prevents most simplifications. Therefore, we instead use a localized variant of persistence pairing that guarantees that each point on the Jacobi set is paired with one of its level set neighbors as described below.

Given a non-degenerate restricted critical point $v \in f_t$ and its two level set neighbors $u, w \in n_g(v)$, the goal is to understand how f_t can be modified in a local neighborhood surrounding v, in order to cancel v with either u or w. Consider, e.g. v_3 shown in Figure 3. One can lower v_3 to the level of v_4 canceling (v_3, v_4), but cannot lower it to the level of v_2 as this would impact v_4, and thus become a non-local simplification. In general, each restricted critical point can be cancelled with only one of its level set neighbors in this fashion, and we call such a neighbor its *partner*. Formally, this relation between a restricted critical point and its partner can be described through a *local pairing function*, $\mu : \mathbb{J} \to \mathbb{J}$, such that for every $v \in \mathbb{J}$, its *partner* $\mu(v)$ is defined as (a) v, if v is a degenerate critical point of f_t or a critical point of g; or (b)
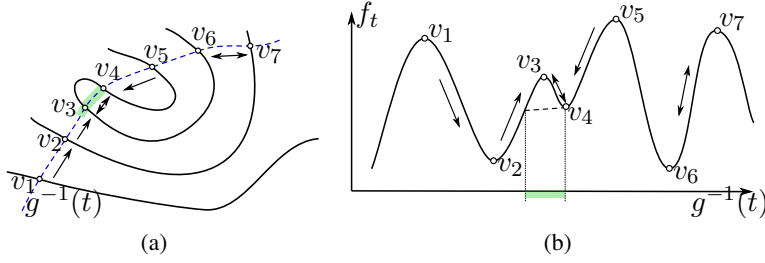
6

Figure 3: Illustration of restricted critical points and local pairings. (a) A Jacobi set $\mathbb{J}$ (black solid lines) intersects a level set $g^{-1}(t)$ (blue dashed line), and (b) the corresponding restricted function f_t is shown. Local pairings among the restricted critical points in f_t are indicated by arrows. The pair (v_3, v_4) can be cancelled by lowering the maximum v_3 to match the value of v_4 (black dashed line in (b)). For the cancellation, its region of influence along the level set is shown in green (in both (a) and (b)). It is the subset of the level set where the function value is modified, however, is shown as a thick region only for illustration.

an arbitrary element in the set $\{u \mid \operatorname{argmin}_{u \in n_g(v)} \|f_t(u) - f_t(v)\|\}$ otherwise. Intuitively, every non-degenerate restricted critical point v is paired with one of its level set neighbors u with minimal difference in function value. Then (v, u) is referred to as a *(local) pair*. Notice that, $\mu(v) = u$ does not imply $\mu(u) = v$. Traveling along a Jacobi curve, the discontinuities of $\mu(v)$ reflect a change in partner for v. Since BD points and extrema of g are paired to themselves, μ is continuous at such points. Figure 3 indicates the pairings between restricted critical points as directed arrows pointing from v to its partner $\mu(v)$.

Using the assigned local pairings between critical points of f_t, a simplification that removes critical points in pairs through atomic cancellation operations is given by the Morse Cancellation Theorem (Theorem 5.4 in [12] as the First Cancellation Theorem, or [13]). We can perform such a cancellation by moving a critical point to the level of its partner to obtain a continuous simplified function $\overline{f_t}$, as shown in Figures 2(a) and 3(b). To obtain a smooth f_t^*, an ϵ-slope can be introduced in $\overline{f_t}$ while still maintaining locality. Consequently, for a pair (v, u), a cancellation where v is moved to the level of u always guarantees locality. Notice in Figure 3(b) that the pair (v_5, v_6) could also be cancelled locally by bringing both points to a function value between v_4 and v_7. In general, one could potentially bring both points to a common intermediate value for a local cancellation. However, such cancellations may not admit valid simplification steps for reasons explained in Section 4.2, and therefore are not considered. From now on, a cancellation induced by a pair (v, u) always implies a procedure that moves v to the level of u.

4.2 Construction and cancellation of Jacobi regions

The cancellation of a pair of restricted critical points creates a smooth restricted function f_t^*. However, the function is still discontinuous across the level set $g^{-1}(t)$, since the neighboring restricted functions are unchanged. Hence, canceling a single pair of restricted critical points in isolation introduces unwanted discontinuities, and therefore violates the smoothness condition of a valid simplification. Instead, one can extend these cancellations to adjacent restricted functions, which, however, violates the consistency condition of a valid simplification. For example, consider the scenario shown in Figure 4. Canceling $(u, v) \in f_{t_0}$ creates a discontinuous simplified function. This modification can be extended to an adjacent region $f_{[t_0 - \epsilon, t_0 + \epsilon]}$ allowing the creation of a smooth function f^* at t_0 which cancels (u, v). However, since $\mathbb{J}$ is now modified beyond the level set $g^{-1}(t_0)$, it is no more a consistent simplification.

Therefore, one must cancel connected sets of neighboring restricted critical points that are paired consistently. To understand their construction, we define *switch points* as the set of points in $\mathbb{J}$ where μ is not
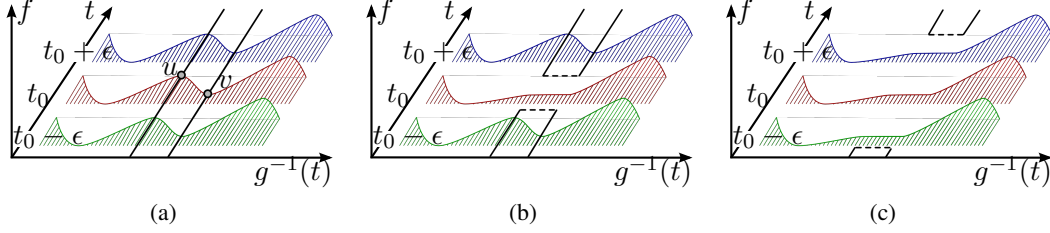
Figure 4: Cancellation of a pair of restricted critical points $(u, v) \in f_t$. (a) The original f_t's and Jacobi set (in black). (b) Canceling (u, v) in f_t in isolation creates a discontinuity across $t = t_0$, and hence is invalid. (c) Extending the cancellation to $f_{[t_0-\epsilon, t_0+\epsilon]}$ creates a smooth f^*, but the cancellation is inconsistent since $\mathbb{J}$ outside $[t_0, t_0]$ is modified.
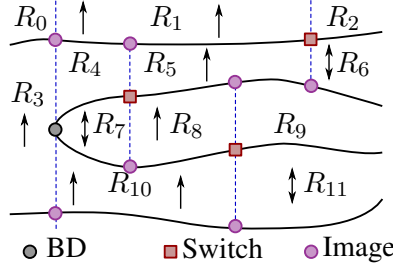

Figure 5: Illustration of Jacobi regions as pairings between Jacobi segments.

continuous, and *boundary points* which are either switch points, BD points, or critical points of g. Then, the Jacobi set $\mathbb{J}$ can be decomposed into a set of non-overlapping *Jacobi segments*, which are maximal open subsets of $\mathbb{J}$ separated by boundary points. By definition, restricted critical points within the interior of Jacobi segments are consistently paired since μ is continuous, and thus μ induces a pairing between segments. Finally, we define *image points* as the level set neighbors of boundary points. Together, the boundary points and the image points decompose the Jacobi set into pieces α_i that have mutually consistent pairing, meaning that μ is continuous both on α_i and its partner $\beta_i = \mu(\alpha_i)$. Given two such maximal subsets of Jacobi segments which are level set neighbors parametrized as $\alpha, \beta : (a, b) \to \mathbb{J}$, we call their bounded region $R_{(a,b)}(\alpha, \beta)$ a *Jacobi region*. Similar to the point-wise cancellation, the entire segment α can be moved to the level of β to cancel both the segments. Figure 5 shows boundary and image points, Jacobi segments, and Jacobi regions as pairings between them for a typical Jacobi set configuration.

There exist various classes of Jacobi regions with different implications on the simplification process. A Jacobi region is called *regular* if its closure does not contain BD points or critical points of g. Regular regions have four "corners" made up of two switch and two image points, e.g. R_5, R_8 in Figure 5. With slight abuse of notation, we denote a corner as $\alpha(a) = \lim_{t \to a} \alpha(t)$. We further identify special but not mutually exclusive types of regions shown in Figure 6: (a) *BD internal regions* where α and β share at least one BD point, i.e. $\alpha(a) = \beta(a)$ and/or $\alpha(b) = \beta(b)$; (b) *BD side region* where α and/or β are bounded by a BD point but $\alpha(x) \neq \beta(x)$, for all $x \in [a, b]$; (c) *BD external region* where the boundary of the region contains a BD point but neither α nor β does; (d) *Saddle region* where the boundary of the region contains a saddle of g but neither α nor β does; and (e) *Extremal region* containing an extremum of g.

By construction, Jacobi segments are paired consistently within each region. Except for extremal regions, which already contain the minimal number of restricted critical points, boundary segments of a Jacobi region $R_{(a,b)}(\alpha, \beta)$ (such that $\mu(\alpha(t)) = \beta(t)$) can be cancelled by setting $\overline{f}_t(\alpha(t)) = f_t(\beta(t))$ for all $t \in (a, b)$. As shown for region R_0 in Figure 7(b), this cancellation will modify f only within a small neighborhood
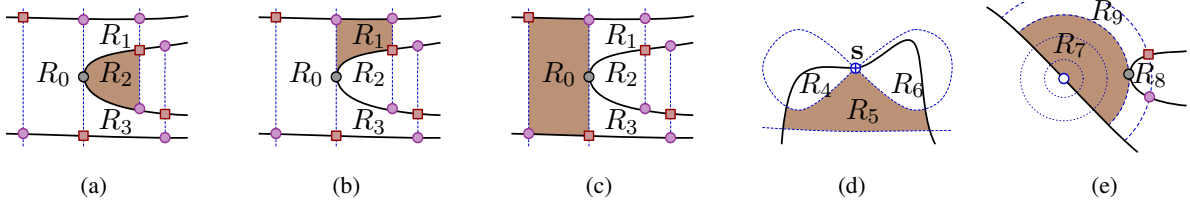
8

Figure 6: Special Jacobi regions: (a) BD internal; (b) BD side; (c) BD external; (d) Saddle; and (e) Extremal region.

around R_0 still bounded by $g^{-1}(t_1)$ and $g^{-1}(t_2)$. We call the modified region as the *region of influence* of the corresponding cancellation and point out that it does not contain portions of $\mathbb{J}$ not part of R_0, and thus satisfies the consistency condition. While this creates a continuous $\overline{f}$ in the region, $\overline{f}$ is still discontinuous at the boundary, and constructing a corresponding smooth f^* requires a non-local change. However, consider the cancellation of R_1 following the cancellation of R_0 as shown in Figure 7(c). By construction, the region of influence of R_1 matches that of R_0 at their shared boundary along $g^{-1}(t_2)$, since $\beta(t_2)$ is a switch point (where $f_t(\alpha(t_2)) = f_t(\gamma(t_2))$). Hence, continuing the cancellation in the obvious manner across $g^{-1}(t_2)$ creates a valid cancellation covering the interval (t_1, t_3). Note that this would not be possible if the cancellation of R_0 modified both α and β, thus we choose to modify the values of either α or β (as pointed out in Section 4.1). In general, given two regular regions $R_{(t_1,t_2)}(\alpha_i, \beta_i)$ and $R_{(t_2,t_3)}(\alpha_j, \beta_j)$ sharing a switch point, there always exists a valid simplification on the interval (t_1, t_3) which removes $\alpha_i, \beta_i, \alpha_j, \beta_j$ as well as their shared switch and image points from $\mathbb{J}$.

In order to obtain a valid $\mathbb{J}(f^*, g)$ consisting of closed loops, the simplification must also reconnect the portions of $\mathbb{J}(f, g)$ rendered disconnected due to the cancellations. For a continuous simplification, this connection can be made within a single restricted function. However, a smooth simplification demands modifications which can not be confined locally. For example, consider the dashed line in Figure 7(c) connecting segments α and γ which shows such a transition. Without loss of generality, assume $\alpha(t)$ and $\gamma(t)$ to be maxima. The corresponding restricted functions in $[t_2 - \epsilon, t_2 + \epsilon]$ are shown in Figure 8. For cancellation of restricted critical points, $\beta(t)$ is moved towards $\alpha(t)$ for $t < t_2$, and towards $\gamma(t)$ for $t > t_2$. Figure 8(b) shows the restricted function when the transition is made with in a single level set. However, to obtain a smooth transition, the simplification must also modify γ in $(t_2 - \epsilon, t_2)$, and α in $(t_2, t_2 + \epsilon)$. As shown in Figure 8(c), the maxima $\gamma(t)$ and $\alpha(t)$ in the corresponding ranges are spatially shifted towards $\beta(t)$ such that $\beta(t_2)$ now becomes a maxima, i.e. $f^*(\beta(t_2)) = f(\alpha(t_2)) = f(\gamma(t_2))$. Since such a transition can always be created at switch points, for simplicity in the following figures, we assume a smooth transition and illustrate them as vertical lines (along a single level set). Although this transition may appear to be a non-local and hence an invalid simplification, we remind the reader that locality is required for continuous
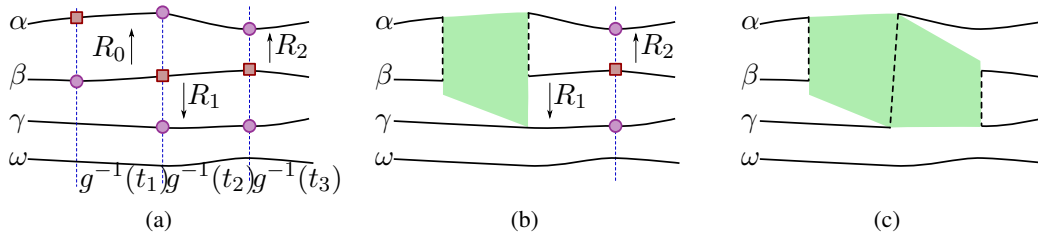

Figure 7: Cancellation of $R_0 = R_{(t_1,t_2)}(\alpha, \beta)$ and $R_1 = R_{(t_2,t_3)}(\beta, \gamma)$ with regions of influence shown in green. The black dashed lines represent the points added to the Jacobi set to connect the existing loops.
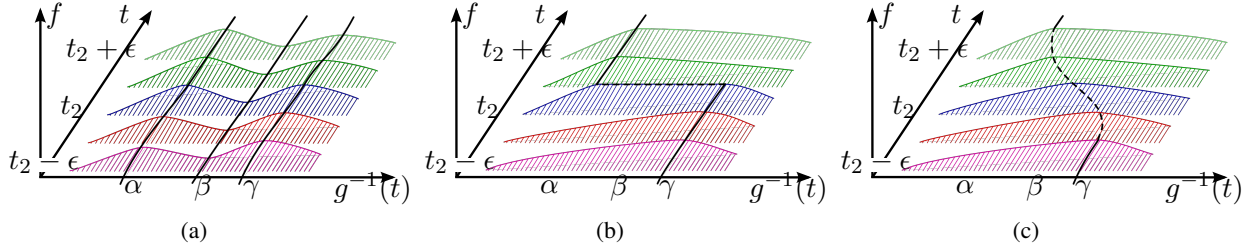
Figure 8: The existing Jacobi curves may be connected smoothly by adding restricted critical points shown along the dashed curve. (a) Zoom-in view of the restricted functions in the neighborhood of $g^{-1}(t_2)$ from Figure 7(a). (b) A continuous simplification can be obtained by creating a transition with in a single level set. (c) However, for a valid simplification, a smooth transition must be made by modifying the restricted functions in $[t_2 - \epsilon, t_2 + \epsilon]$.

simplification only. Furthermore, since the Jacobi set remains unchanged outside of $[t_1, t_3]$, consistency is maintained, therefore, producing a valid simplification.

4.3 Construction and cancellation of Jacobi sequences

As discussed above, one can construct (partially) valid simplifications by simultaneously canceling adjacent Jacobi regions. In this section, we describe how to assemble *Jacobi sequences* as ordered sets of regions that allow a valid simplification. Formally, we call two Jacobi regions *adjacent* if they share a boundary point, and we use the function value of g to induce an ordering among adjacent regions. To construct a sequence that admits a valid simplification, it is important to understand (a) where such a sequence may start or end; and (b) how to construct its corresponding simplification $\overline{f}$ and ultimately f^*.

Following the discussion in Section 4.2 we claim that valid sequences are naturally bounded by BD internal regions. This is because at the BD point, the region of influence shrinks to a single point and any arbitrary small interval outside the BD point allows the construction of a smooth f^*. More specifically, consider a sequence of Jacobi regions covering the interval (a, b) that starts and ends with BD internal regions, and contains only regular regions otherwise. Given the discussion above, for any $\epsilon > 0$ we can create a smooth f^* covering the interval $(a - \epsilon, b + \epsilon)$ which cancels all restricted critical points in the closure of the sequence. By construction f^* is local, smooth, and consistent, and thus forms a valid simplification.

Further, we note that BD external, BD side, extremal, and saddle regions can never be part of a valid simplification. Refer to Figure 6 and notice that it is not possible to continue across the BD point for BD external and BD side regions, since the discontinuity across the level set of BD point can not be removed locally. Similar argument holds for a saddle region, whose cancellation leaves unresolvable discontinuity around the saddle. Finally, an extremal region cannot be cancelled since all the level sets inside the region contain only two restricted critical points, and cannot be simplified further.

As a result, valid sequences are comprised of only regular regions and BD internal regions, where they must begin and end with a BD internal region. Therefore, all sequences are seeded at BD internal regions and constructed by progression into adjacent regions monotonically in g until another BD internal region is encountered, at which point the sequence is considered complete. Due to the ordering imposed on adjacent regions, a sequence cannot form loops.

On the other hand, if during its construction, a sequence encounters any of the regions that can not be simplified, it is considered invalid and discarded. Although such regions can invalidate some sequences, this does not stop the simplification from progressing. If no valid sequence exists due to the presence of saddle and/or extremal regions, we perform a conventional 2D critical point cancellation in g to create new
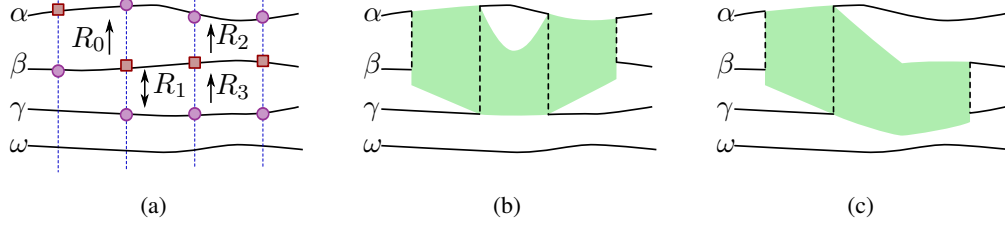
Figure 9: Mutually-paired regions offer a choice of the segment to be moved. (a) Original configuration, where R_0 can be cancelled by moving β towards α. Subsequently, R_1 can be cancelled by: (b) moving β to γ leading to the sequence $\{R_0, R_1, R_2\}$. (c) smoothly transitioning between moving β to moving γ leading to sequence $\{R_0, R_1, R_3\}$.
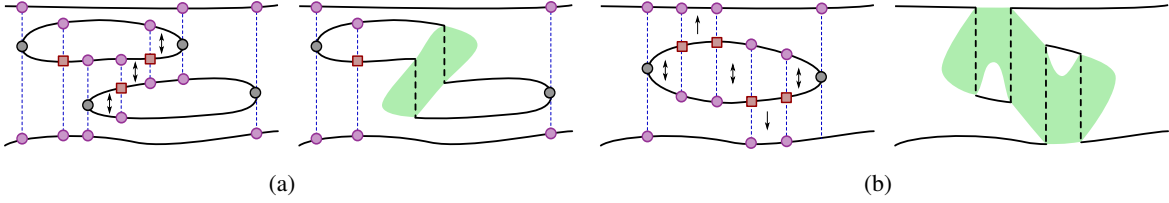

Figure 10: Our simplification algorithm performing (a) merging and (b) removal of loops. Pairings are shown only for the regions that are cancelled, and the corresponding regions of influence for the simplification.

sequences. This cancellation does not change the Jacobi set structurally, and can be done independent of any sequence cancellation. Section 5 discusses saddle cancellation in detail. Again, the BD external or BD side regions may invalidate some sequences. However, in such a case, we can always seed a new sequence from the corresponding BD internal region.

From Section 4.2, we know that a region $R_{(a,b)}(\alpha, \beta)$, such that $\mu(\alpha(t)) = \beta(t)$ for all $t \in (a, b)$, can be cancelled by moving the segment α to the level of β, that is, by setting $\overline{f}(\alpha(t)) = f(\beta(t))$. However, if the region is *mutually paired*, meaning $\mu(\alpha(t)) = \beta(t)$ and $\mu(\beta(t)) = \alpha(t)$, one can move either α or β. This provides flexibility in sequence construction, as one can smoothly transition from moving α to moving β. Since valid simplification requires cancellation of adjacent regions in which the same segment can be moved to its respective partners, it follows that one can potentially cancel either of the two adjacent regions after canceling R. For example, consider Figure 9 where regions R_0 and R_1 are already a part of a Jacobi sequence. For cancellation in R_0, the segment β is moved to match the value of α. For cancellation in R_1, we can either continue moving β towards γ, or switch segments by smoothly transitioning from moving β to moving γ. The former leads to the sequence $\{R_0, R_1, R_2\}$ where β is moved to its respective partners in all regions, while the latter leads to $\{R_0, R_1, R_3\}$ where β and γ are moved in R_0 and R_3 respectively, while a transition between moving β and moving γ is performed in R_1.

We point out that the critical points of f are naturally cancelled as critical points of f_t. Further, the construction and cancellation of Jacobi sequences can handle general form of structural changes to the Jacobi set. As examples, we show the merging and removal of loops from Jacobi set in Figure 10.

4.4 Ordering the cancellations

In order to obtain a hierarchy on the simplification process, we need to define a metric to measure the amount of modification needed for each simplification step. Although the choice of the metric is flexible,

we choose a gradient-based metric capable of measuring the relative variation between the two functions inside a region, i.e. the *comparison measure* $\kappa(R)$ (see Section 2). Our choice is inspired by the fact that the cancellation of a region creates a *flat f^** in its interior, i.e. $\|\nabla f^*\| \le \epsilon$. An alternative formulation of κ [15] by rewriting it as an integral over the Jacobi set is

$$\kappa(R) = \frac{1}{2\text{Area}(R)} \int_{v \in \mathbb{J}} |2f(v) - f(u) - f(w)| \cdot \|\nabla g(v)\| \, dv, \tag{3}$$

where, $u, w \in n_g(v)$. Therefore, $\kappa(R)$ for every region R can be computed by integrating over its bounding segments. The modification needed to cancel a Jacobi sequence, is the sum of modifications of all regions in the sequence. Similarly, $\kappa(R)$ can also be defined for affected regions in the critical point cancellation (see Section 5). Thus, all valid simplification steps can be performed in the increasing order of κ.

Construction of simplified function f^*   Given a Jacobi sequence $\{R_i\}$ spanning the level sets in $[a, b]$, the steps to construct the simplified functions can be summarized below.

Step 1. For all the regions included in the sequence, a continuous function $\overline{f}$ is created by canceling corresponding Jacobi segments, as guided by the pairing function. This modification is local and is achieved by "flattening" the function to an appropriate value.

Step 2. A smooth simplified function f^* is created by following modifications.

- To obtain smoothness along level sets, $\overline{f}$ is perturbed to induce an ϵ_1-slope along the level sets, using any $\epsilon_1 > 0$ (e.g. see Figure 2(a)).
- To obtain smoothness across level sets while maintaining consistency, $\overline{f}$ is perturbed in the range $(a - \epsilon_2, a]$ and $[b, b + \epsilon_2)$, using any $\epsilon_2 > 0$ (e.g. see Figure 2(b)).
- To obtain smoothness in transitions between existing Jacobi curves, restricted critical points are spatially shifted in an ϵ_3-neighborhood of level sets, using any $\epsilon_3 > 0$. (e.g. see Figure 8(c)).

5   Cancellation of critical points in g

As discussed in Section 4.3, no valid simplification sequence of f can cancel a critical point of g. However, there may exist configurations such that all Jacobi sequences of f contain critical points of g and all sequences in g contain critical points in f. In this case, there exists no valid sequence and the Jacobi set cannot be simplified through a standard cancellation. Instead, we use traditional critical point cancellations to remove pairs of critical points from either function. In this section, we show that critical points on g can be cancelled with minimal impact to the geometry of $\mathbb{J}(f, g)$. Furthermore, we describe the change in κ caused by such a cancellation, and how the pairing among Jacobi segments is affected. Rather than the typical persistence pairing, we use a slightly relaxed notion of critical point pairing centered around the notion of an *isolated* pair.

Definition 5.1 *A saddle-maximum* $(\mathbf{s}, \mathbf{m})$ *pair of a function g is called* isolated *if the component of the super-level set $g^{-1}(g(\mathbf{s}))$ containing $\mathbf{m}$ does not contain any other critical points of g.*

Clearly, on a simply-connected domain all extrema of g except for a single maximum and minimum can be removed through successive cancellation of isolated pairs. The section is divided into four parts: First, we describe how given an isolated maximum-saddle pair $(\mathbf{s}, \mathbf{m}_1)$ in g, one can construct a smooth function g^* that only differs from g in a arbitrary small neighborhood of the super-level set around $\mathbf{m}_1$; Second, we show that $\mathbb{J}(f, g) = \mathbb{J}(f, g^*)$ except for an ϵ-neighborhood around $\mathbf{s}$; and Third, we discuss how the cancellation affects Jacobi segments and regions; and last, we identify the modification needed for this cancellation.
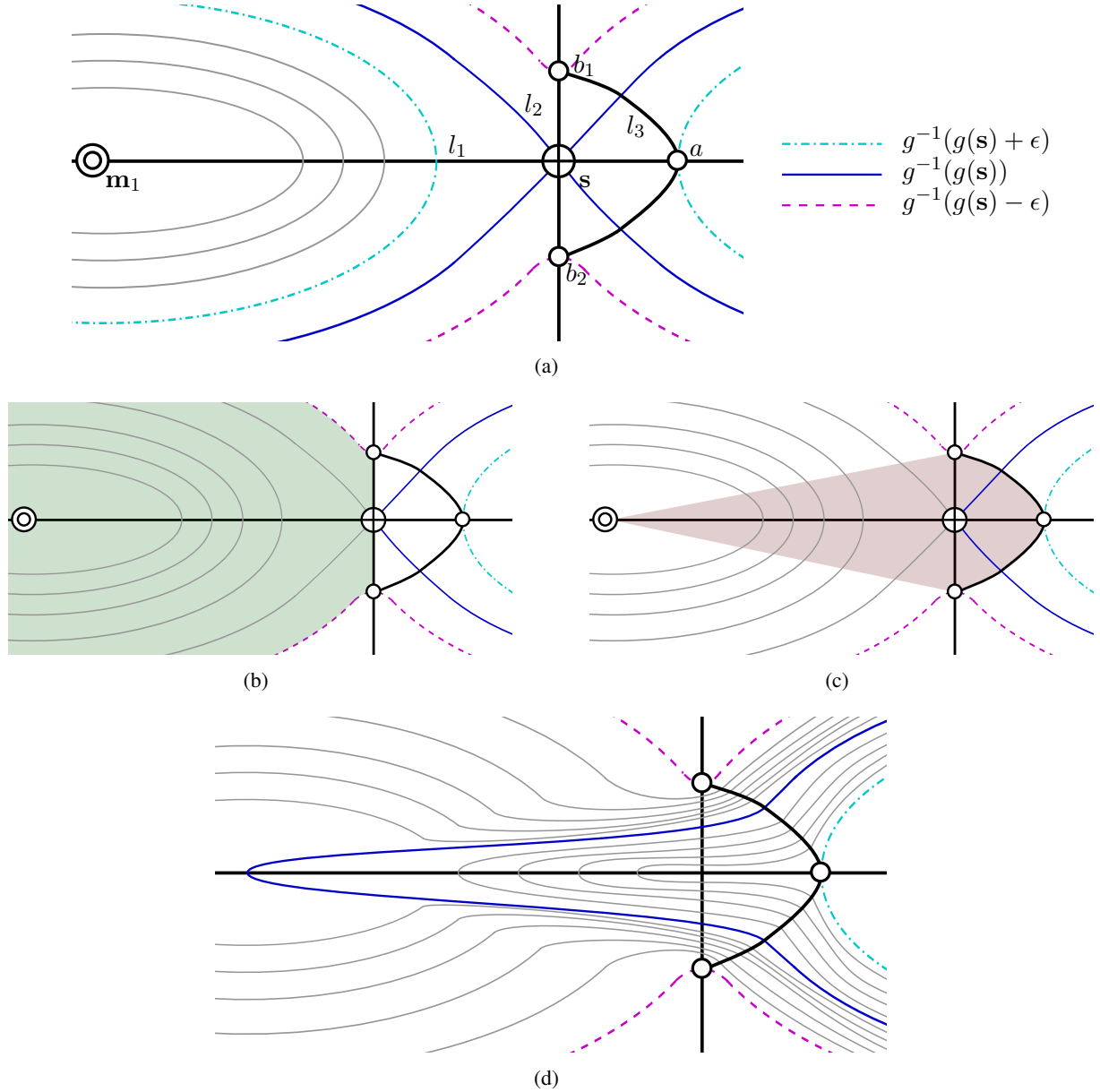
Figure 11: Cancellation of a saddle-maximum pair, $(\mathbf{s}, \mathbf{m}_1)$. (a) Level sets of the original function, g. Line l_1 connects $\mathbf{m}_1$ with a $\left(\in g^{-1}(g(\mathbf{s}) + \epsilon)\right)$ through $\mathbf{s}$. Line l_2 connects b_1 and b_2 with $b_i \in g^{-1}(g(\mathbf{s}) - \epsilon)$. Line l_3 connects b_1, a, and b_2. (b) The shaded region bounded by $g^{-1}(g(\mathbf{s}) - \epsilon)$ and l_2 is rescaled to the range $[g(\mathbf{s}) - \epsilon, g(\mathbf{s})]$. This creates a discontinuity along l_2. (c) The shaded region C is identified and g is modified such that ∇g^* is anti-parallel to ∇g along l_1, and $\nabla g^*(x) = \nabla g(x)$ for all $x \in \partial C$. This cancels the $(\mathbf{s}, \mathbf{m}_1)$ pair, and removes the discontinuity in 11(b). (d) The final function, g^* does not contain the saddle $\mathbf{s}$, and the maximum $\mathbf{m}_1$. The modification (from g to g^*) is confined only to the shaded regions in Figures 11(b) and 11(c).

13

Critical points pair cancellation   Consider the canonical level set structure of g around $\mathbf{s}$ shown in Figure 11(a) and the three level sets $g^{-1}(g(\mathbf{s}))$, $g^{-1}(g(\mathbf{s}) + \epsilon)$, and $g^{-1}(g(\mathbf{s}) - \epsilon)$. Furthermore, consider three lines: l_1, connecting $\mathbf{m}_1$ with $\mathbf{s}$ and a point a on $g^{-1}(g(\mathbf{s}) + \epsilon)$; l_2, connecting $\mathbf{s}$ with two points b_1 and b_2 on $g^{-1}(g(\mathbf{s}) - \epsilon)$; and l_3 connecting b_1, a, b_2 as shown in Figure 11(a).

To create a g^* that cancels $(\mathbf{s}, \mathbf{m}_1)$ we first rescale g within the super-level set of $g^{-1}(g(\mathbf{s}) - \epsilon)$ on the left of l_2 (Figure 11(b)) by monotonically mapping the range $[g(\mathbf{s}) - \epsilon, g(\mathbf{m}_1)]$ to the range $[g(\mathbf{s}) - \epsilon, g(\mathbf{s})]$. Choosing the appropriate map, the resulting function $\bar{g}$ is smooth except at l_2 where it is discontinuous. Note that $\bar{g}(\mathbf{m}_1) = g(\mathbf{s})$. Subsequently, we modify $\bar{g}$ to be monotonically increasing along l_1. Finally, we define a cone C around l_1 as a region enclosed by l_3 and containing $\mathbf{m}_1$ (Figure 11(c)). Now we define a smooth g^* with $g^* = \bar{g}$ outside C, $\nabla g^*(x) = \nabla g(x)$ for all $x \in \partial C$, and ∇g^* is anti-parallel to ∇g for all $x \in l_1$, as shown in Figure 11(d). Such a g^* exists for all $\epsilon > 0$ as the solution of a boundary value problem.
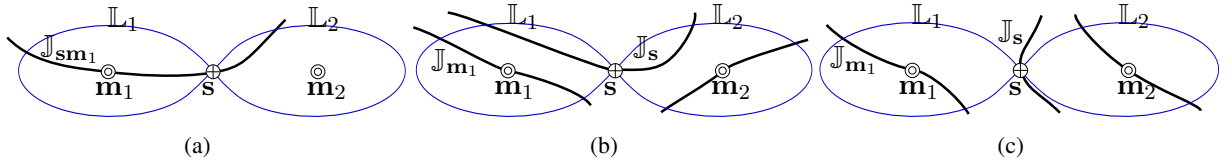

(a)   (b)   (c)

Figure 12: Different cases of Jacobi set connectivity for an isolated saddle-maximum pair $(\mathbf{s}, \mathbf{m}_1)$. The saddle and the maximum may be parts of by (a) the same Jacobi set component $\mathbb{J}_{\mathbf{s},\mathbf{m}_1}$, or (b) and (c) separate Jacobi components $\mathbb{J}_\mathbf{s}$ and $\mathbb{J}_{\mathbf{m}_1}$ respectively. $\mathbb{L}_1$ and $\mathbb{L}_2$ are super level sets of g surrounding $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively.

Jacobi set geometry   Generically, there exist three different configurations of Jacobi sets in the neighborhood of an isolated critical point pair as shown in Figure 12. The most common configuration is a Jacobi set connecting $\mathbf{m}_1$ and $\mathbf{s}$ (Figure 12(a)). In this case we can define $l_1 \subset \mathbb{J}$ and l_2 as a subset of the unstable 1-manifold of $\mathbf{s}$ (i.e. gradient descending path of a saddle) which guarantees that $\nabla g^*(x)$ and $\nabla g(x)$ are aligned, for all $x \in \mathbb{J}(f, g)$. It follows that $x \in \mathbb{J}(f, g)$ implies $x \in \mathbb{J}(f, g^*)$. Furthermore, for all $x \notin C$ we have $\nabla g^*(x) = \nabla g(x)$ which implies $\mathbb{J}(f, g) = \mathbb{J}(f, g^*)$ for $\mathbb{M} \setminus C$. However, with g^* as defined above, there may exist additional points $x \in C$ with $x \in \mathbb{J}(f, g^*)$. By construction, these must be part of isolated Jacobi components (loops) entirely contained in C. As such they must form a valid cancellation sequence and can be remove using the approach discussed in Section 3.

The situations shown in Figures 12(b) and 12(c) follow a similar argument except that it cannot be guaranteed that $\mathbb{J}(f, g) = \mathbb{J}(f, g^*)$ around $\mathbf{s}$. However, since this portion of the Jacobi set enters and exists C exactly once, there must exist a g^* that connects the entry and exit points with a single line of the Jacobi set containing no BD points. Therefore, $\mathbb{J}(f, g) \neq \mathbb{J}(f, g^*)$ only in a small neighborhood around $\mathbf{s}$.

Modifications in Jacobi segments and Jacobi regions   To understand how the Jacobi segments and Jacobi regions are affected by this cancellation, we refer to Figure 13. In addition to the lines and points described above (in Figure 11), let c_1 and c_2 be the points where l_3 intersects with $g^{-1}(g(\mathbf{s}))$, and d the point of intersection of l_1 with $g^{-1}(g(\mathbf{s}) - \epsilon)$. The cancellation modifies the shape of the level sets in specific ways shown in Figure 11(d). That is, each point $x \in \overline{d\mathbf{m}_1}$ (i.e. the line between d and $\mathbf{m}_1$) is connected to some $y \in \overline{b_1 c_1}$ on one side (of $g^{-1}(\mathbf{s})$), and some $z \in \overline{b_2 c_2}$ on the other side. Similarly, each $x \in \overline{\mathbf{m}_1 a}$ is connected either to some $y \in \overline{c_1 a}$, or some $z \in \overline{c_2 a}$. The pairings in f_t must be recomputed along the modified level sets, and new regions need to be created.

To illustrate the modifications in pairings and Jacobi regions, we give an example of such cancellation in Figure 14. The figure shows that most of the regions are unaffected. Only the regions that included the cone

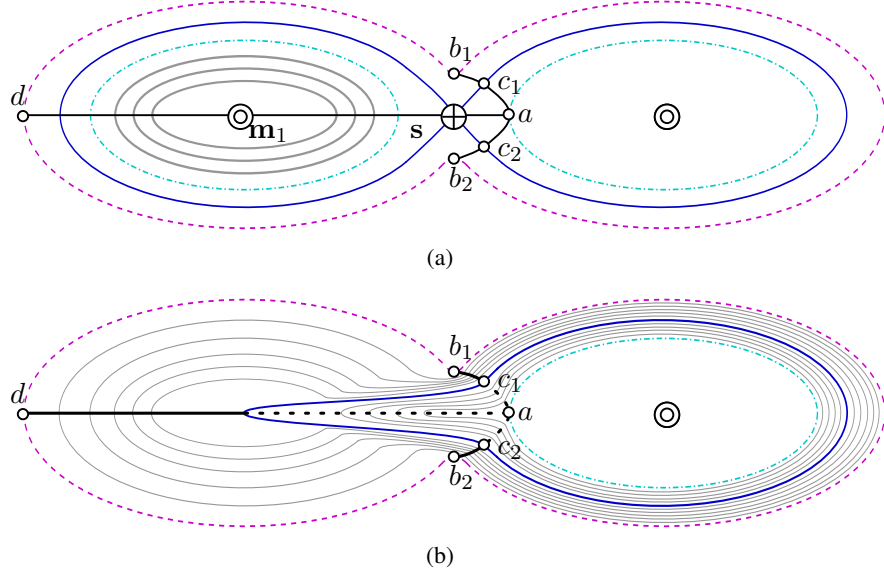C as described above are modified, and extend along the new level sets.

(a)

(b)

Figure 13: The modifications in level sets due to the cancellation of $(\mathbf{s}, \mathbf{m}_1)$, as shown (a) before and (b) after the cancellation.
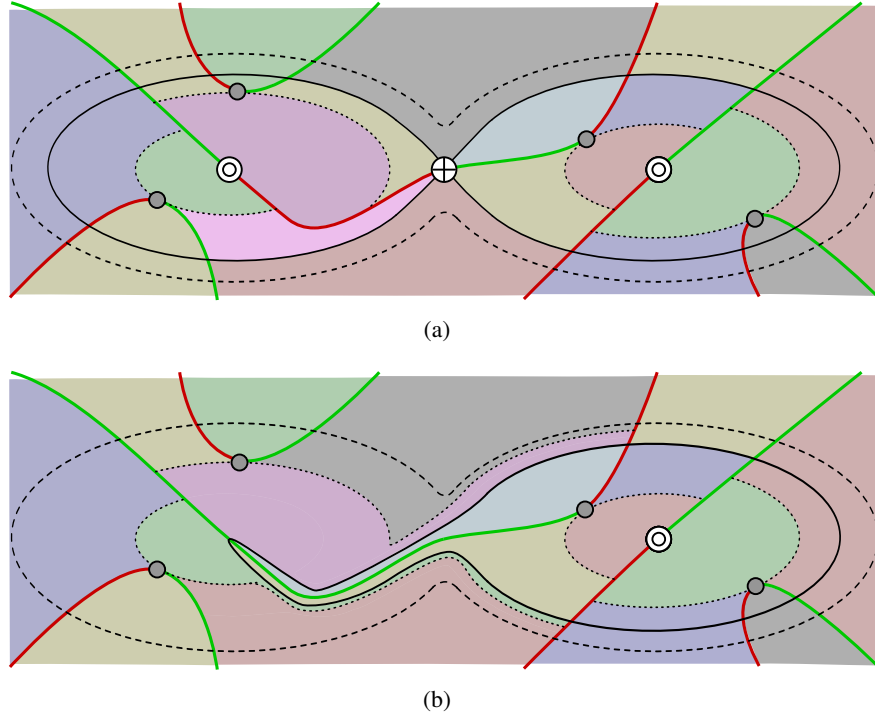

(a)

(b)

Figure 14: Effect of saddle cancellation on Jacobi segments and Jacobi regions. In addition to the level sets of BD points (dotted), level sets $g^{-1}(g(\mathbf{s}))$ (solid) and $g^{-1}(g(\mathbf{s}) - \epsilon)$ (dashed) are shown for reference. The Jacobi set is shown as red-green curve, with color representing the criticality (i.e. restricted maximum or minimum). Jacobi regions corresponding to (a) the original Jacobi set $\mathbb{J}(f, g)$, and (b) the Jacobi set after cancellation, $\mathbb{J}(f, g^*)$, are shown in different colors.

15

Modification needed for the cancellation We note that $|\nabla g^*(x)| = O(\epsilon)$ for all $x \in \mathbb{L}_1$ (where $\mathbb{L}_1$ is the super level set surrounding $\mathbf{m}_1$). Then, the comparison measure of $\mathbb{L}_1$ after cancellation, κ^*, is given by

$$\kappa^*(\mathbb{L}_1) = \frac{\int_{\mathbb{L}_1} \|\nabla f(x) \times \nabla g^*(x)\| \, \mathrm{d}x}{\mathrm{Area}\,(\mathbb{L}_1)} = O(\epsilon)$$

Note that κ^* is independent of both the difference in the function values of $\mathbf{s}$ and $\mathbf{m}_1$, and the shape of $\mathbb{L}_1$. Thus, the amount of perturbation introduced by this cancellation is approximately $\lim_{\epsilon \to 0}(\kappa - \kappa^*) = \kappa$.

6 Summary and Correctness

Given the discussion on the cancellation of restricted critical points of f_t, and critical points of g, we now summarize the complete procedure to simplify a given Jacobi set. So far, all the discussion has focused on modifying f with respect to the level sets of g. However, we may wish to interleave the modifications of either of these functions with respect to the other. Thus, to simplify the Jacobi set, we need to identify all Jacobi sequences with respect to both – the level sets of f and the level sets of g.

Step 1. Identify all possible simplification steps with respect to the level sets of g/f, by creating all possible Jacobi sequences, and identifying all isolated saddle-extremum pairs.

- Compute the pairings between restricted critical points and identify the switch points.
- Create Jacobi segments by decomposing $\mathbb{J}$ into subsets bound by the BD points in $\mathbb{J}$, critical points of g/f, switch points, and their images.
- Create Jacobi regions using the pairings induced on the segments, and compute their κ.
- Create Jacobi sequences $\{S\}_g$ and $\{S\}_f$ by seeding them at BD internal regions, and propagating monotonically into adjacent regions in a depth-first manner, and compute its κ.
- Identify all isolated saddle-extremum pairs $\{P\}_g$ and $\{P\}_f$, and compute their κ.

Step 2. Store all sequences $\{S\}_f$ and $\{S\}_g$, and all pairs of saddle-extremum pairs $\{P\}_f$ and $\{P\}_g$ into a common list $\mathcal{L}$, ordered by their κ, the amount of modification needed for their cancellation.

Step 3. Select the element (S or P) with the lowest κ from $\mathcal{L}$, perform its cancellation, recompute the pairings in $\mathbb{J}$, and create corresponding Jacobi regions.

Step 4. Remove from $\mathcal{L}$ all the existing sequences that cease to exist due to this cancellation, and identify and add to $\mathcal{L}$ any new sequences containing the newly created regions.

Step 5. Repeat steps 3 and 4 until the Jacobi set reaches its simplest possible configuration under our definition of validity or a user-defined threshold is achieved.

Correctness and Termination In order to prove the correctness of this simplification scheme, we note that by definition, every valid simplification step ensures that the resulting function f^* (or g^*) is Morse, and the simplified Jacobi set reflects the Jacobi set of the simplified functions. Therefore, we have the following corollary:

Corollary 6.1 *The simplified functions f^* and g^* are Morse and the simplified Jacobi set is a valid Jacobi set $\mathbb{J}(f^*, g^*)$.*

By construction, the algorithm terminates when no other pair of restricted critical points can be cancelled through a valid simplification, and no other isolated critical points can be cancelled through conventional critical point cancellation.

16

In Section 6.1, we study the minimal Jacobi set configuration possible for a given domain. Here, the minimal configuration means that the functions f and g are Morse functions with the minimal number of critical points, and the Jacobi set $\mathbb{J}(f, g)$ has the minimal number of loops and contains no BD points. Section 6.2 shows that for simply connected domains, our algorithm achieves this minimal configuration. Finally, Section 6.3 discusses the challenges in removing certain kinds of Jacobi segments on non-simply connected domains. We discuss configurations with more than the minimal number of loops which cannot be simplified through local modification, but an extensive study on handling such cases is beyond the scope of this paper.

6.1 Minimal Jacobi sets

As with most simplification procedures, the goal is to reach the simplest possible configuration according to some measure. In the case of Jacobi sets, for a pair of Morse functions f and g defined on a smooth, compact, and orientable 2-manifold $\mathbb{M}$ without boundary, the goal is to reduce the number of loops and BD points in $\mathbb{J}(f, g)$. We focus on Jacobi set with minimal configuration, i.e. the functions f and g are Morse functions with the minimal number of critical points, the Jacobi set $\mathbb{J}(f, g)$ has the minimal number of loops and contains no BD points. Previously, Bennett et al. [2] suggested that given a domain with genus γ the minimal Jacobi set has $\gamma + 1$ loops. We disprove this claim by showing that there exist functions f and g on $\mathbb{M}$ that give rise to at least one and at most two Jacobi loops. Especially for manifold with even genus, a Jacobi set with one loop exists. Furthermore, for $\gamma = 0$ we prove that our algorithm will reach the minimal configuration. Unfortunately, for $\gamma > 1$ there exist configurations with more than the minimal number of loops which cannot be simplified through local modification.

To proof the lemma below, we first give a construction of a Jacobi set containing two loops on a (single) torus, and a single loop on a double-torus. Since a manifold of even genus is homeomorphic to a connected sum of double-tori, and a manifold of odd genus is homeomorphic to a connected sum of double-tori and a (single) torus, we can apply a similar construction procedure to show that there exist functions f and g such that $\mathbb{J}(f, g)$ has a single loop for even genus and two loops for an odd genus.

Lemma 6.1 *The minimal Jacobi set $\mathbb{J}(f, g)$ on a manifold $\mathbb{M}$ of genus γ contains at least 1 and at most 2 loops.*

Proof. In the case when $\gamma = 0$, it is easy to see that there exist f and g that create only a single loop. For example, imaging a sphere embedded into $\mathbb{R}^3$ centred at the origin. Then two height functions f and g with 90 degree angle, that is, $f(x, y, z) = x$ and $g(x, y, z) = z$ will create such a Jacobi set.

For $\gamma > 0$, $\mathbb{M}$ is homeomorphic to a connected sum of γ tori. Such a surface can be constructed as the union of bent and straight cylinders as shown in Figure 15. Imaging each piece embedded into $\mathbb{R}^3$ with $g(x, y, z) = z$, the height function. Defining $f(x, y, z) = x$ creates a Jacobi set that follows the silhouette and creates $\gamma + 1$ loops. However, along a straight cylinder we can smoothly transition to $f(x, y, z) = -x$ (and the reverse) which winds the Jacobi set around the cylinder in a half turn (Figure 15(e)-(f)). Combining these twisted cylinders one can reconnect the default $\gamma + 1$ loops. The Jacobi sets for the torus and double-torus are also shown in Figure 16 without the gluing cylinders.

In particular, as shown in Figure 15(h) for a double-torus we can connect all pieces into a single loop. Clearly, as shown Figure 17, by combining double-tori this creates functions with a single Jacobi loop for all surfaces with even genus. However, for a single torus the same technique simply intertwines two loops (Figure 15(g)). Nevertheless, treating a surface with odd genus as one with even genus plus a torus, there must exist f and g that create only two loops which proves the lemma. □

We conjecture that for surfaces with uneven genus two loops is the minimal configuration as the re-combinations must come in pairs but currently there exists no proof.
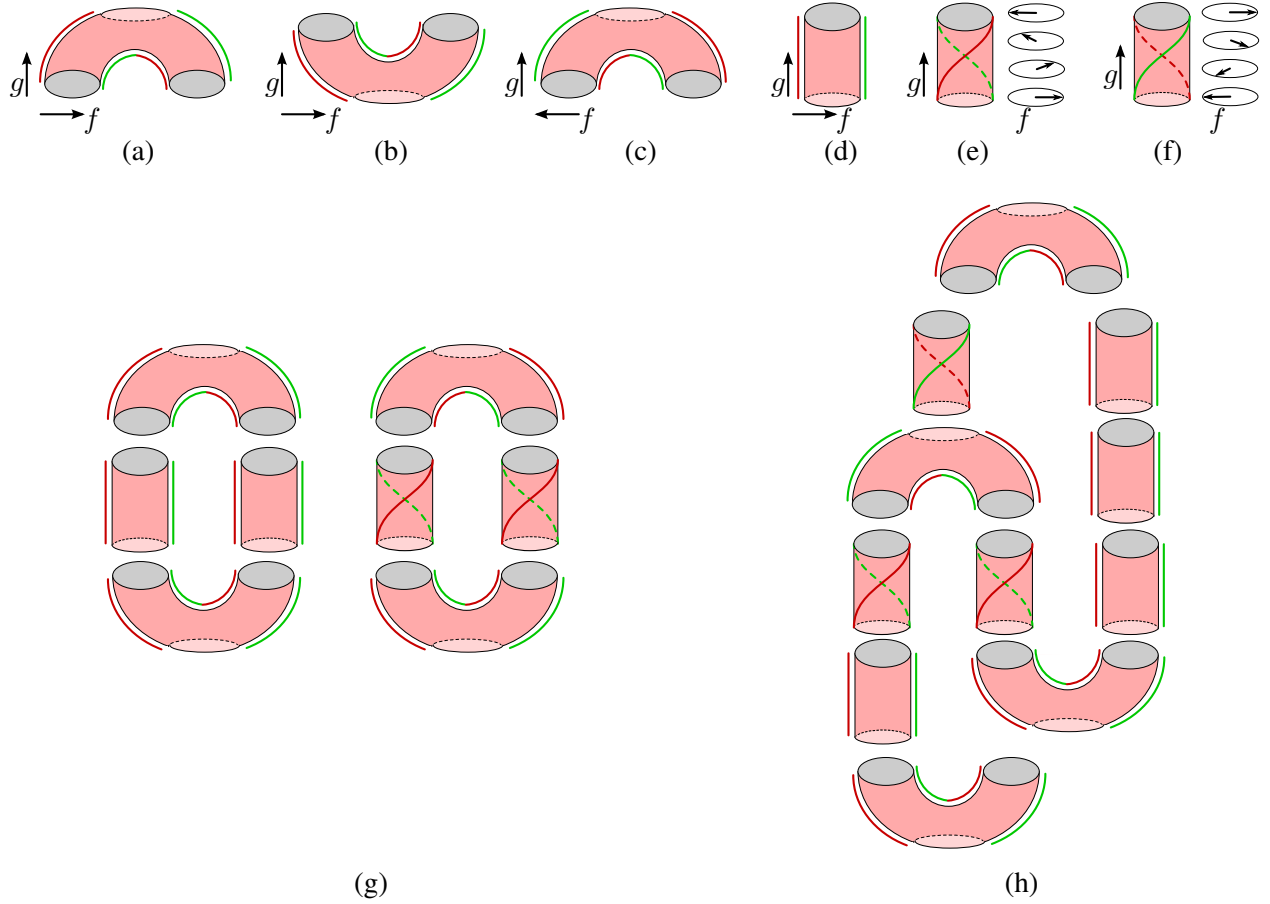
Figure 15: A single (g) and double-torus (h) can be constructed by gluing together six smaller pieces (a) to (f). The arrows indicate the directions of functions f and g, the colors of the Jacobi loops denote criticality of f_t, and the dashed lines denote part of the loop on the back side of the manifold. In (e) and (f), f is smoothly changed to $-f$ from top to bottom. This operation rotates the Jacobi loop between left and right of the corresponding pieces. When the pieces are glued together, this rotation makes it possible for a single Jacobi loop to connect all the critical points of f and g for a double-torus. However, for a single torus, it simply interchanges the connectivity of the two loops.

6.2   Simplification of Jacobi set on simply connected domains

To show that our simplification can achieve the minimal configuration on simply connected domains, we first argue that if two BD points are connected by a Jacobi loop, there always exists a valid sequence that removes both BD points from the Jacobi set. Furthermore, assuming g contains only two extrema on a simply connected domain, there exist only a single configuration (shown in Figure 18) such that the Jacobi set contains BD points not connected by the same Jacobi loop. Subsequently, we prove that these BD points must also be connected by a valid sequence.

Lemma 6.2 *If $\mathbb{M}$ is a simply connected domain, and u, v are two BD points connected by a Jacobi loop such that no critical points of g or other BD points are between them. Then, there exists a sequence of Jacobi regions connecting u with v that forms a valid simplification.*

Proof. Let $t_1, t_2 \in \mathbb{R}$ denote the function values of g at the BD points, that is, $t_1 = g^{-1}(u)$ and $t_2 = g^{-1}(v)$, and without loss of generality, assume $t_1 < t_2$. The BD points create and destroy two restricted critical
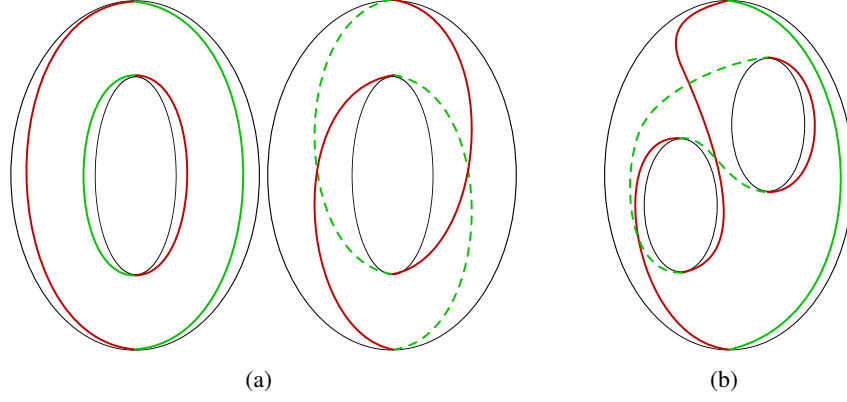
Figure 16: (a), (b) The Jacobi set on a single torus ($\gamma = 1$) contains 2 loops, with two possible configurations. (c) In the case of a double torus ($\gamma = 2$), a configuration with a single Jacobi loop is feasible. The color of the Jacobi loops denote criticality of f_t, the dashed line denotes the loop on the back side of the torus.
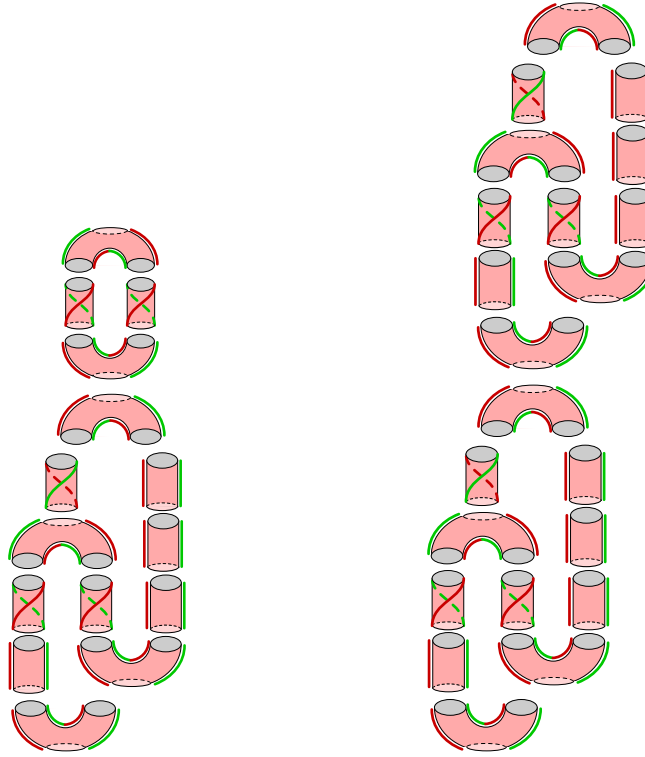

Figure 17: A three-torus (left) constructed as connected sum of a single torus and a double-torus ($\mathbb{T}^3 = \mathbb{T}\#\mathbb{T}^2$), and a four-torus (right) constructed as connected sum of two double-tori ($\mathbb{T}^4 = \mathbb{T}^2\#\mathbb{T}^2$). Clearly, all critical points on the four-torus can be connected by the single loop. On the other hand, for a three-torus, one needs two Jacobi loops.

points. Since the restricted functions $f_{t_0-\epsilon}$ and $f_{t_1+\epsilon}$ are Morse and hence must contain at least two restricted critical points, it follows that for all $t \in (t_0, t_1)$, f_t has at least four restricted critical points. As a result, each point on the Jacobi set connecting u with v is paired and can be cancelled with its partner. Since at a BD point, $\mathbb{J}$ is always mutually paired on the "inside" (of the BD internal region) there must exist a valid
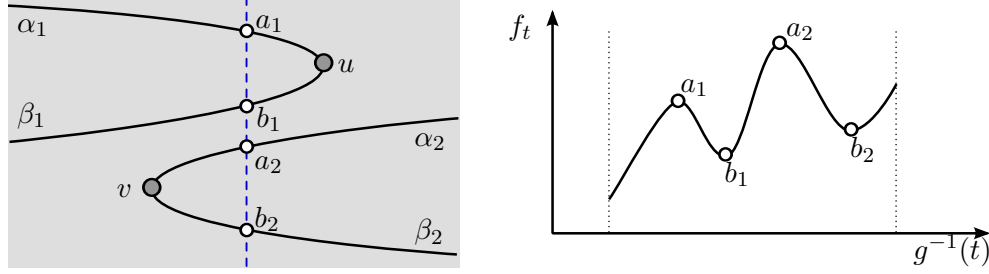
Figure 18: A pair of BD points must always be connected by a Jacobi sequence. Consider a level set $g^{-1}(t)$ between the level sets containing the BD points u and v. Since the corresponding f_t is periodic, both pairs of restricted critical points (a_1, b_1), and (a_2, b_2) can not be mutually paired. Thus, there must exist a region $R_{[t_1,t_2]}(\alpha_1, \beta_2)$, or $R_{[t_1,t_2]}(\alpha_2, \beta_1)$ leading to a valid sequence connecting u and v.

sequence or Jacobi regions connecting u with v.

Note that the two possible configurations for this scenario are shown in Figures 2(b) and 10(b). □

To prove the main result we note that on a simply connected domain all saddles of f and g can be removed either through simplifying the Jacobi set or through direct cancellations. That is, on a simply connected domain all saddles of f and g can be removed through cancellations and only a single minimum and a single maximum remains. In this case, any potentially remaining BD points must form a valid sequence of regions, as no critical points exist that may block a sequence from being formed.

Lemma 6.3 *If $\mathbb{M}$ is a simply connected domain ($\gamma = 0$), the algorithm reduces a Jacobi set to its minimal configuration – a single loop without birth death points.*

Proof. Without loss of generality, we suppose f and g contain no saddles, since for simply connected domain, all saddles can be cancelled (either through sequence cancellation or by direct cancellation with extrema of the functions). As a result the level sets of g can be seen as a collection of vertical lines periodic at ∞ as shown in Figure 18. Following Lemma 6.2, all BD points connected by Jacobi loops can be removed through valid cancellations. Nevertheless, there can exist two BD points are not connected to each other, one forming a loop with the maximum of g and one forming a loop with the minimum of g. Both loops must overlap since each f_t must have at least two restricted critical points.

Assume there does not exist a sequence connecting the two BD points. It follows that the curve α_1 is always mutually paired with β_1 and α_2 is always mutually paired with β_2. We show that this is a contradiction. Assume a_1, a_2 are maxima and b_1, b_2 are minima. If α_1 is mutually paired with β_1 then $f(b_1) > f(b_2)$ and $f(a_1) < f(a_2)$. (Remember that the level sets are periodic.) However, α_2 mutually paired with β_2 implies $f(a_2) < f(a_1)$ which gives a contradiction, and hence proves the lemma. □

6.3 Simplification of Jacobi set on non-simply connected domains

On simply connected domains, we showed that our simplification scheme can obtain the minimal Jacobi set configuration. Here, we discuss the fundamental problems that are inherent to the structure of non-simply connected domains, and how they impact our simplification algorithm. The simplification algorithm terminates when no more valid simplification steps are possible, and hence no more Jacobi sequences can be found. For non-simply connected domains, there exist saddles that can not be removed. These saddles may block the construction of Jacobi sequences, and hence no more Jacobi sequences may be formed. Thus, our algorithm may terminate without achieving the minimal Jacobi set.

To contrive such an example, we start with the minimal Jacobi set (shown in Figure 16(a)) on a torus $\mathbb{T}$, where the two functions are height functions with an angle $\pi/2$ between them. The function f can then be changed along the outer silhouette of the torus, using a sinusoidal kernel that replaces the restricted maxima with a valley and restricted minima with a ridge. For each f_t, this operation replaces one restricted critical point by three, thus creating two extra Jacobi loops. Since the function must stay smooth, the kernel must go to zero at the critical points of g, where the restricted critical points of f_t switch.

To understand this Jacobi set, recall that the torus is constructed as the product of two circles. If θ and ϕ denote the polar angle of the two circles, then the torus can be parametrized as $\mathbb{T}(\theta, \phi)$. Figure 19 shows the level sets and critical points of the two functions (in red and blue) on the $\theta - \phi$ plane along with the Jacobi set (in black). Clearly, there exist four loops in the Jacobi set. The saddles on $\mathbb{J}_1$ and $\mathbb{J}_3$ also act as BD points. Any sequences that are seeded at the BD points always get stuck at the saddles and hence, no valid sequence is possible.

Notice that in this case, there exists no simplification that can be achieved by local modification to the functions. Thus, our algorithm can not simplify this Jacobi set further. Going forward, we envision more general and global simplifications steps, which modify more than two loops of Jacobi set simultaneously. Such simplifications will be able to handle such difficult cases for non-simply connected domains.
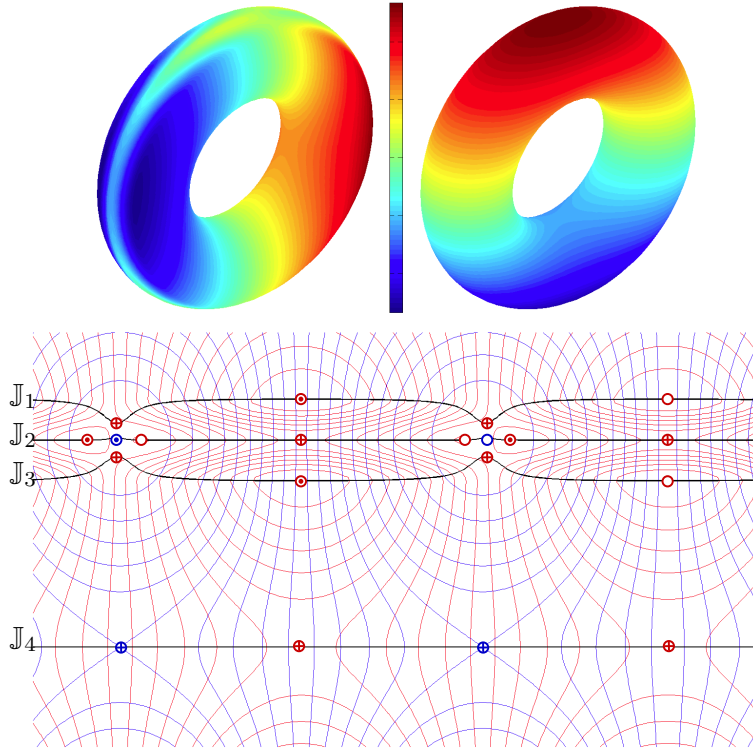

Figure 19: (Top) Functions f (left) and g (right) are defined on a torus, $\mathbb{T}(\theta, \phi)$. (Bottom) The level sets and critical points of f and g are shown in red and blue respectively, along with the Jacobi set in black, on the $\theta - \phi$ plane. Since the domain is periodic, the four Jacobi loops are closed. Although, there exist BD points on $\mathbb{J}_3$ and $\mathbb{J}_4$ (coinciding with saddles of f), the algorithm can not find a valid Jacobi sequence due to the presence of irremovable saddles.

7 Discussion and Future Work

In this paper, we introduce a direct technique for Jacobi set simplification, aimed at achieving local, smooth, and consistent modifications to the underlying functions. Our technique is guided by a user-defined metric, and offers fine control over the simplification process and could be widely applicable in many data analysis applications. While our choice of the metric – the comparison measure κ is well-suited for our purpose, we would like to explore other possibilities. Further, while it is clear that the algorithm reduces a Jacobi set to its simplest configuration for simply connected domains, there exist cases where this is not possible for non-simply connected domains. Understanding of such cases may lead to the need of global simplification operations which can help obtain the simplest Jacobi set for manifolds with non-zero genus. We wish to explore such cases and extend our simplification scheme to address them. Lastly, while the focus of the current work is a detailed discussion on the various elements of the algorithm for smooth functions, a technical discussion on its discrete adaptation and practical implementation is forthcoming.

Acknowledgement

References

[1] T. P. Barnett, D. W. Pierce, and R. Schnur. Detection of climate change in the world's oceans. *Science*, 292:270–274, 2001.

[2] J. Bennett, V. Pascucci, and K. Joy. Genus oblivious cross parameterization: Robust topological management of inter-surface maps. *Proceedings Pacific Graphics*, pages 238–247, 2007.

[3] P.-T. Bremer, E. M. Bringa, M. A. Duchaineau, A. Gyulassy, D. Laney, A. Mascarenhas, and V. Pascucci. Topological feature extraction and tracking. *Journal of Physics: Conference Series*, 78:012007, 2007.

[4] H. Edelsbrunner and J. Harer. Jacobi sets of multiple Morse functions. In F. Cucker, R. DeVore, P. Olver, and E. Süli, editors, *Foundations of Computational Mathematics, Minneapolis 2002*, pages 37–57. Cambridge University Press, 2002.

[5] H. Edelsbrunner, J. Harer, A. Mascarenhas, V. Pascucci, and J. Snoeyink. Time-varying Reeb graphs for continuous space–time data. *Computational Geometry: Theory and Applications*, 41:149–166, 2008.

[6] H. Edelsbrunner, J. Harer, V. Natarajan, and V. Pascucci. Local and global comparison of continuous functions. In *Proceedings 16th IEEE Conference on Visualization*, pages 275–280, 2004.

[7] H. Edelsbrunner, D. Letscher, and A. J. Zomorodian. Topological persistence and simplification. *Discrete and Computational Geometry*, 28:511–533, 2002.

[8] H. Edelsbrunner, D. Morozov, and A. Patel. The stability of the apparent contour of an orientable 2-manifold. In V. Pascucci, X. Tricoche, H. Hagen, and J. Tierny, editors, *Topological Methods in Data Analysis and Visualization: Theory, Algorithms, and Applications*, pages 27–42. Springer-Verlag, 2009.

[9] C. Luo, I. Safa, and Y. Wang. Approximating gradients for meshes and point cloud via diffusion metric. In *Proceedings Symposium on Geometry Processing*, pages 1497–1508, Aire-La-Ville, Switzerland, 2009. Eurographics Association.

[10] Y. Matsumoto. *An Introduction to Morse Theory*. American Mathematical Society, 2002. Translated from Japanese by Kiki Hudson and Masahico Saito.

[11] J. Milnor. *Morse Theory*. Princeton University Press, 1963.

[12] J. Milnor. *Lectures on the h-cobordism theorem, Princeton Mathematical Notes*. Princeton University Press, 1965.

[13] M. Morse. Bowls of a non-degenerate function on a compact differentiable manifold. *Differential and Computational Topology (A Symposium in Honor of Marston Morse)*, 75:81–104, 1965.

[14] S. N and V. Natarajan. Simplification of Jacobi sets. In V. Pascucci, X. Tricoche, H. Hagen, and J. Tierny, editors, *Topological Data Analysis and Visualization: Theory, Algorithms and Applications*, pages 91–102. Springer-Verlag, 2011.

[15] V. Natarajan. *Topological analysis of scalar functions for scientific data visualization*. PhD thesis, Department of Computer Science, Duke University, 2004.

[16] G. Norgard and P.-T. Bremer. Ridge-valley graphs: Combinatorial ridge detection using Jacobi sets. *Computer Aided Geometric Design*, 2012.

[17] Y. Shinagawa and T. L. Kunii. Constructing a Reeb graph automatically from cross sections. *IEEE Computer Graphics and Applications*, 11(6):44–51, 1991.

[18] V. G. Szebehely and H. Mark. *Adventures in Celestial Mechanics*. Wiley, New York, second edition, 1998.

[19] A. J. Zomorodian and G. Carlsson. Computing persistent homology. *Discrete and Computational Geometry*, 33:249–274, 2005.